Study of some orthosymplectic Springer fibers

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February 24, 2010

Abstract

We decompose the fibers of the Springer resolution for the odd nilcone of the Lie superalgebra $\mathfrak{osp}(2n+1, 2n)$ into locally closed subsets. We use this decomposition to prove that almost all fibers are connected. However, in contrast with the classical Springer fibers, we prove that the fibers can be disconnected and non equidimensional.

Introduction

As for classical Lie algebras, the odd nilpotent cone $\mathcal{N}_1$ of the Lie superalgebra $\mathfrak{osp}(2n+1, 2n)$ has a natural resolution (cf. [GrLe09] and section 1). We call it the *Springer resolution* and denote it by $\pi : \widetilde{\mathcal{N}_1} \to \mathcal{N}_1$. The purpose of the present paper is to describe some properties of the fibers of π. On the one hand, some results, true for Lie algebras, are no longer true in the Lie superalgebra setting. Indeed, the study of explicit examples leads to the following proposition.

Proposition 0.1. *The fibers of π are, in general, neither connected nor equidimensional. In particular, the variety $\mathcal{N}_1$ is not normal.*

On the other hand we give for the orthosymplectic Lie superalgebra, as in the Lie algebra setting, a decomposition of the fiber of π into locally closed subsets (see Theorem 1.12). These subsets do not have the same dimension in general and their closures are not always irreducible components of the Springer fiber. We use this decomposition to prove the following result.

Theorem 0.2. *Let $\mathcal{O}_1$ be the unique codimension 1 orbit in $\mathcal{N}_1$ and let $X \in \mathcal{N}_1$. The fiber $\pi^{-1}(X)$ is connected if and only if $X \notin \mathcal{O}_1$.*

As an application of our decomposition, we also describe explicitely the fibers of π above the three bigger non dense orbits of $\mathcal{N}_1$ under the action of the orthosymplectic group. For the unique codimension 3 orbit, the fiber is not equidimensional and has irreducible components of dimension 1 and 2 (see Proposition 3.3).

1 Decomposition of the fiber

1.1 Some basic facts about odd nilpotent orbits

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space of super-dimension $(2n+1, 2n)$ and equipped with a bilinear super-symmetric form B. This means that the spaces V_0 and V_1 are orthogonal, of respective dimensions $2n+1$ and $2n$ and the restrictions $\varphi : V_0 \to V_0^\vee$ and $\psi : V_1 \to V_1^\vee$ of B to V_0 and V_1 are non degenerate respectively symmetric and alternate. Let $\mathfrak{osp}(2n+1, 2n)$ be the Lie superalgebra, called the orthosymplectic Lie superalgebra, consisting in endomorphisms of V which preserve the bilinear super-symmetric form and the graduation (for more details, see for example [GrLe09]).

The group $\mathrm{O}(V_0,\varphi)\times\mathrm{Sp}(V_1,\psi)$ acts on $\mathfrak{osp}(2n+1,2n)$. It is called the orthosymplectic group and we denote it by $G_0\times G_1$. Let u be an element of $\mathrm{Hom}(V_0,V_1)$, we define $u^*\in\mathrm{Hom}(V_1,V_0)$ by $u^*=\varphi^{-1}\circ u^t\circ\psi$. An endomorphism $X=(u,u^*)$ of V, with $u\in\mathrm{Hom}(V_0,V_1)$ is called odd orthosymplectic and belongs to $\mathfrak{osp}(2n+1,2n)$. Any degree 1 element in $\mathfrak{osp}(2n+1,2n)$ preserving the $\mathbb{Z}/2\mathbb{Z}$-graduation is of this shape.

The set of odd nilpotent orthosymplectic endomorphisms of $\mathrm{End}(V)$ is a cone denoted by $\mathcal{N}_1$. A $G_0\times G_1$-equivariant resolution of the singularities of $\mathcal{N}_1$ is constructed in [GrLe09]. Let us describe this resolution.

Let B_0 and B_1 be Borel subgroups of G_0 and G_1. For $X=(u,u^*)\in\mathcal{N}_1$ we denote by $B_u=B_X$ the set of pairs of isotropic complete flags $((E_i)_{i\in[1,n]},(F_j)_{j\in[1,n]})$ in $G_0/B_0\times G_1/B_1$, with $E_i\subset V_0$, $\dim E_i=i$, $F_j\subset V_1$, $\dim F_j=j$ and for all $i\in[1,n]$

$$X(E_i)=u(E_i)\subset F_{i-1}\text{ and }X(F_i)=u^*(F_i)\subset E_i. \tag{†}$$

By analogy with the classical case, the variety $B_u=B_X$ corresponding to $X=(u,u^*)\in\mathcal{N}_1$ is called Springer fiber since it can be identified with the fiber above X of the resolution $\widetilde{\mathcal{N}_1}$ of the singularities of $\mathcal{N}_1$:

$$\widetilde{\mathcal{N}_1}=\{(u,(E_i)_{i\in[1,n]},(F_j)_{j\in[1,n]})\in\mathrm{Hom}(V_0,V_1)\times G_0/B_0\times G_1/B_1\ /\ ((E_i),(F_j))\in B_u\}.$$

The map $\pi:\widetilde{\mathcal{N}_1}\to\mathcal{N}_1$ is the first projection, the second one p_2 realizes $\widetilde{\mathcal{N}_1}$ as a vector bundle above $G_0/B_0\times G_1/B_1$.

H.P. Kraft and C. Procesi [KrPr82] proved that the odd nilpotent orthosymplectic orbits under $G_0\times G_1$-action of $\mathfrak{osp}(m,2n)$ (where $m\in\{2n+2,2n+1,2n,2n-1,2n-2\}$) are parametrized by marked Young diagrams of size $m+n$ (see Fulton [Ful97] for more details on Young diagrams). We recall their results and specify that the diagrams are written in the french way.

Definition 1.1. *(ı) A* marked diagram *of size (m,n) is a Young diagram of size $m+n$ in which there are m boxes labelled with* 0 *and n boxes labelled with* 1*. The labels in the same line alternate.*

(ıı) A line beginning with $\epsilon\in\{0,1\}$ is said to be of parity ϵ.

(ııı) A marked diagram D is called indecomposable *if it has one of the following shapes:*

1. *an even line of length $4p+1$,*
2. *an odd line of length $4p-1$,*
3. *two even lines of length $4p-1$,*
4. *two odd lines of length $4p+1$,*
5. *two lines, one even, one odd of length $2p$.*

(ıv) A marked diagram is admissible *if it is the union of indecomposable diagrams.*

Proposition 1.2. *[KrPr82] There is a bijective correspondence between odd nilpotent orbits of $\mathfrak{osp}(m,2n)$ and admissible diagrams of size (m,n).*

An easy consequence of the above correspondence is the following fact.

Fact 1.3. *Let $X\in\mathcal{N}_1$, D be its associated diagram by the previous correspondence. The dimension of the space $KerX\cap\mathrm{Im}X^{k-1}\cap V_\epsilon$ is the number of lines of parity ϵ and of length at least k. The super form is trivial on this space except for $k=1$. In this case its rank is the number of marked lines of parity ϵ and length 1.*

1.2 Slicing of diagrams

An admissible diagram D is said to have a parity if its size is $(2n + (-1)^\epsilon, 2n)$ for some n and $\epsilon \in \{0, 1\}$; the parity of D is then ϵ. (If D is a line of the given size this coincides with the previous notion of parity.)

Definition 1.4. *Let D be an admissible diagram having a parity. We call* admissible subdiagram *of D any subdiagram D' of D such that*

1. *D' is admissible,*
2. *D' has two boxes less than D,*
3. *D' and D have different parities,*
4. *the boxes of $D \setminus D'$ are at the beginning and at the end of lines of same length.*

Lemma 1.5. *Let D be an admissible diagram of parity ϵ and k be an integer such that D has at least one line of length k and parity ϵ. Then there exist at least one and at most two admissible subdiagrams D' of D such that the boxes in $D \setminus D'$ lie on lines of length k.*

Proof : The two boxes are removed either on the same line or on two different lines. The different cases are as follows.

• If k is even, the admissibility of D and D' (definition 1.1-(5)) implies that the number of lines of length k of D and D' is even and then two lines of length k change size i.e. the set $D \setminus D'$ is on two different lines.

• If $k = 4p + (-1)^\epsilon$, the admissibility of D and D' (definition 1.1-(1)-(2)) implies that we can have any number of lines of length k in D and of lines of length $k - 2$ in D'. We can choose $D \setminus D'$ on the same line or on two different lines if there exist two lines of length k.

• If $k = 4p - (-1)^\epsilon$, the admissibility of D and D' (definition 1.1-(3)-(4)) implies that the number of lines of length k in D has to be even and then that two lines of length k change size i.e. the set $D \setminus D'$ is on two different lines. $\square$

Definition 1.6. *Let D be an admissible diagram of size $(2n + 1, 2n)$ and $D_0 = D$. A sequence $(D_i)_{i \in [0,2n]}$ such that D_i is a admissible subdiagram of D_{i-1} for $i \in [1, 2n]$, is called an* admissible slicing *of D and is denoted by $\underline{D}$. We denote by $\mathcal{A}(D)$ the set of admissible slicings of D.*

Example 1.7. *We give below the unique two admissible slicings of a diagram.*

$$\begin{array}{lllll} 0 \\ 1&0&1 \\ 0&1&0&1&0 \end{array} \supset \begin{array}{lll} 0 \\ 1&0&1 \\ 1&0&1 \end{array} \supset \begin{array}{lll} 0 \\ 0 \\ 1&0&1 \end{array} \supset \begin{array}{lll} 1&0&1 \end{array} \supset \begin{array}{l} 0 \end{array}$$

$$\begin{array}{lllll} 0 \\ 1&0&1 \\ 0&1&0&1&0 \end{array} \supset \begin{array}{lll} 0 \\ 1&0&1 \\ 1&0&1 \end{array} \supset \begin{array}{ll} 0 \\ 1&0 \\ 0&1 \end{array} \supset \begin{array}{l} 0 \\ 1 \\ 1 \end{array} \supset \begin{array}{l} 0 \end{array}.$$

1.3 Locally closed subsets in the fiber

Let X be a nilpotent element in $\mathfrak{osp}(2n + (-1)^\epsilon, 2n)$. We denote by $\mathcal{K}_\epsilon(X, k)$ the set of isotropic points of $\mathbb{P}(V_\epsilon \cap \mathrm{Ker}X \cap \mathrm{Im}X^{k-1}) \setminus \mathbb{P}(V_\epsilon \cap \mathrm{Ker}X \cap \mathrm{Im}X^k)$ and by $\ell_\epsilon(X, k)$ the number of lines of length k and parity ϵ in the diagram associated to X.

Remark 1.8. *($\imath$) When $\epsilon = 1$ or when $\epsilon = 0$ and $\ell_0(X, 1) = 0$, all the points of the projective space $\mathbb{P}(V_\epsilon \cap \mathrm{Ker}X \cap \mathrm{Im}X^{k-1})$ are isotropic.*

(ii) The variety $\mathcal{K}_0(X,1)$ is empty when $\ell_0(X,1)=1$. For the other cases we have

$$\dim \mathcal{K}_\epsilon(X,k) = \begin{cases} \sum_{i\geq k} \ell_\epsilon(X,i) - 2 & \text{if } \epsilon=0,\ k=1 \text{ and } \ell_0(X,1)>1 \\ \sum_{i\geq k} \ell_\epsilon(X,i) - 1 & \text{otherwise.} \end{cases}$$

(iii)The variety $\mathcal{K}_\epsilon(X,k)$ is irreducible (therefore connected) except when $\epsilon=0$, $k=1$ and $l_0(X,1)=2$. In this last case, the isotropic locus is the union of two hyperplanes H and H' in $\mathbb{P}(\mathrm{Ker}X)$ and we have $\mathcal{K}_0(X,1)=(H\cup H')\setminus(H\cap H')$.

Example 1.9. *Let X be a nilpotent element in $\mathfrak{osp}(5,4)$ with associated diagram as follows:*

$$D = \begin{array}{|c|c|ccc} \hline 0 & 1 & \multicolumn{3}{c}{} \\ \hline 1 & 0 & \multicolumn{3}{c}{} \\ \hline 0 & 1 & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} \\ \hline \end{array}\ .$$

Its kernel has dimension 3 and its intersection with V_0 is totally isotropic of dimension 2. We have then $\dim \mathcal{K}_0(X,2)=1$ *and* $\dim \mathcal{K}_0(X,5)=0$.

Proposition 1.10. *Let $X\in\mathfrak{osp}(2n+(-1)^\epsilon,2n)$ be a nilpotent element with associated diagram D, let $x\in\mathcal{K}_\epsilon(X,k)$ and let y be such that $X^{k-1}(y)=x$. The restriction $X|_{x^\perp/x}$ of X to $x^\perp/x$ lies in $\mathfrak{osp}(2n-1,2n-2\epsilon)$ and its orbit under the corresponding orthosymplectic group is associated to the admissible diagram*

- *obtained by removing two boxes in a line of length k of D if $B(x,y)\neq 0$,*
- *obtained by removing two boxes in two different lines of length k of D if $B(x,y)=0$.*

Proof : We first notice that if X is an odd orthosymplectic nilpotent element, its diagram is entirely determined by the dimensions of $\mathrm{Ker}X^a\cap V_0$ and $\mathrm{Ker}X^a\cap V_1$ for all a. We therefore have to compute $\dim(\mathrm{Ker}(X|^a_{x^\perp/x}))$.

We first determine $\mathrm{Ker}(X^a)\cap x^\perp$. If $z\in\mathrm{Ker}X^a$ with $a\leq k-1$, we have the equality $B(x,z)=B(X^{k-1}(y),z)=B(y,X^{k-1}(z))=0$ i.e. $\mathrm{Ker}X^a\subset x^\perp$ for $a\leq k-1$. But $\mathrm{Ker}X^k\not\subset x^\perp$ otherwise we get $(\mathrm{Im}X^k)^\perp=\mathrm{Ker}X^k\subset x^\perp$ and $x\in\mathrm{Im}X^k$, a contradiction with $x\in\mathcal{K}_\epsilon(X,k)$. Recall that $x\in V_\epsilon$ we thus obtain the equalities:

$$\dim(\mathrm{Ker}X^a\cap x^\perp\cap V_\epsilon) = \begin{cases} \dim(\mathrm{Ker}X^a\cap V_\epsilon) & \text{if } a\leq k-1 \\ \dim(\mathrm{Ker}X^a\cap V_\epsilon)-1 & \text{if } a>k-1, \end{cases}$$

$$\dim(\mathrm{Ker}X^a\cap x^\perp\cap V_{1-\epsilon}) = \dim(\mathrm{Ker}X^a\cap V_{1-\epsilon}) \text{ for all } a\in\mathbb{N}.$$

Let us consider $Y=X|_{x^\perp}$ which is possible since $\mathrm{Im}X=\mathrm{Ker}X^\perp\subset x^\perp$. We compute the dimensions of $\mathrm{Ker}Y^a=\mathrm{Ker}X^a\cap x^\perp$. Set $Z=X|_{x^\perp/x}$. We compute $\dim\mathrm{Ker}Z^a$ using $\dim\mathrm{Ker}Y^a$. By definition, we have $\mathrm{Ker}Z^a=(Y^a)^{-1}(\langle x\rangle)/x$, we obtain

$$\dim((Y^a)^{-1}(\langle x\rangle)) = \begin{cases} \dim\mathrm{Ker}X^a+1 & \text{si } a\leq k-1 \\ \dim\mathrm{Ker}X^a & \text{si } a>k-1. \end{cases}$$

If $a\leq k-1$, the element $X^{k-1-a}(y)$ is in $(Y^a)^{-1}(\langle x\rangle)$ but not in $\mathrm{Ker}X^a$. Therefore we have the equalities:

$$\dim((Y^a)^{-1}(\langle x\rangle)\cap x^\perp) = \begin{cases} \dim\mathrm{Ker}X^a+1 & \text{if } a<k-1, \\ \dim\mathrm{Ker}X^a+1 & \text{if } a=k-1 \text{ and } B(x,y)=0, \\ \dim\mathrm{Ker}X^a & \text{if } a=k-1 \text{ and } B(x,y)\neq 0, \\ \dim\mathrm{Ker}X^a & \text{si } a>k-1. \end{cases}$$

We have $\dim \mathrm{Ker} Z^a = \dim((Y^a)^{-1}(\langle x\rangle) \cap x^\perp) - 1$ and for $x \in V_\epsilon$, we have the inclusion $(Y^a)^{-1}(\langle x\rangle) \subset V_{\epsilon+a \mod 2}$. We obtain then the following dimensions.

$$\dim(\mathrm{Ker}Z^a \cap V_\epsilon) = \begin{cases} \dim(\mathrm{Ker}X^a \cap V_\epsilon) - 1 & \text{if } a < k-1 \text{ and } a \text{ is odd,} \\ \dim(\mathrm{Ker}X^a \cap V_\epsilon) & \text{if } a < k-1 \text{ and } a \text{ is even,} \\ \dim(\mathrm{Ker}X^a \cap V_\epsilon) - 1 & \text{sif } a = k-1,\ k-1 \text{ is odd and } B(x,y) = 0, \\ \dim(\mathrm{Ker}X^a \cap V_\epsilon) & \text{if } a = k-1,\ k-1 \text{ is even and } B(x,y) = 0, \\ \dim(\mathrm{Ker}X^a \cap V_\epsilon) - 1 & \text{if } a = k-1 \text{ and } B(x,y) \neq 0, \\ \dim(\mathrm{Ker}X^a \cap V_\epsilon) - 1 & \text{if } a > k-1. \end{cases}$$

$$\dim(\mathrm{Ker}Z^a \cap V_{1-\epsilon}) = \begin{cases} \dim(\mathrm{Ker}X^a \cap V_{1-\epsilon}) + 1 & \text{if } a < k-1 \text{ and } a \text{ is odd,} \\ \dim(\mathrm{Ker}X^a \cap V_{1-\epsilon}) & \text{if } a < k-1 \text{ and } a \text{ is even,} \\ \dim(\mathrm{Ker}X^a \cap V_{1-\epsilon}) + 1 & \text{if } a = k-1,\ k-1 \text{ is odd and } B(x,y) = 0, \\ \dim(\mathrm{Ker}X^a \cap V_{1-\epsilon}) & \text{if } a = k-1,\ k-1 \text{ is even and } B(x,y) = 0, \\ \dim(\mathrm{Ker}X^a \cap V_{1-\epsilon}) & \text{if } a = k-1 \text{ and } B(x,y) \neq 0, \\ \dim(\mathrm{Ker}X^a \cap V_{1-\epsilon}) & \text{if } a > k-1. \end{cases}$$

The diagrams determined by these dimensions are the required diagrams. □

Let X be an odd nilpotent element in $\mathfrak{osp}(2n+1, 2n)$ and let $D(X)$ be its associated diagram. Let $\underline{D} = (D_i)_{i\in[0,2n]}$ be an admissible slicing of $D(X)$. We denote by X_{2i-1} (resp. X_{2i}) the restriction of X to $(E_i^\perp/E_i) \cap (F_{i-1}^\perp/F_{i-1})$ (resp. to $(E_i^\perp/E_i) \cap (F_i^\perp/F_i)$).

Definition 1.11. *We define the subset $B_X(\underline{D})$ of the fiber (X, B_X) by*

$$B_X(\underline{D}) = \{((E_i)_{i\in[1,n]}, (F_j)_{j\in[1,n]}) \in p_2(\pi^{-1}(X)) \ / \ D(X_{2i-1}) = D_{2i-1} \text{ and } D(X_{2i}) = D_{2i}\}.$$

Theorem 1.12. *Let X be an odd nilpotent element in $\mathfrak{osp}(2n+1, 2n)$. The subsets $B_X(\underline{D})$ are locally closed in B_X and we have*

$$B_X = \coprod_{\underline{D}\in\mathcal{A}(D(X))} B_X(\underline{D}).$$

Proof : Let X be an odd nilpotent element in $\mathfrak{osp}(2n+1, 2n)$, we proceed by induction on the size of the diagram $D = D(X)$. Denote by $\mathcal{F}_k$ and $\mathcal{F}'_k$ the varieties

$$\mathcal{F}_k = \{((E_i)_{i\in[1,k]}, (F_j)_{j\in[1,k]}) \ / \ E_i \text{ and } F_j \text{ satisfy the equation } (\dagger)\} \text{ and}$$
$$\mathcal{F}'_k = \{((E_i)_{i\in[1,k]}, (F_j)_{j\in[1,k-1]}) \ / \ E_i \text{ and } F_j \text{ satisfy the equation } (\dagger)\}.$$

We have a sequence of morphisms $\mathcal{F}_n \to \mathcal{F}'_n \to \mathcal{F}_{n-1} \to \cdots \mathcal{F}_1 \to \mathcal{F}'_1 \to \mathcal{F}_0 = \{\mathrm{pt}\}$. The fiber of the morphism $\mathcal{F}'_i \to \mathcal{F}_{i-1}$ (resp. $\mathcal{F}_i \to \mathcal{F}'_i$) is given by the isotropic elements of $\mathbb{P}(\mathrm{Ker}Y)$ where Y is the restriction of X to $(E_{i-1} \oplus F_{i-1})^\perp/(E_{i-1} \oplus F_{i-1})$ (resp. to $(E_i \oplus F_{i-1})^\perp/(E_i \oplus F_{i-1})$). Those Y are orthosymplectic and their associated diagrams are as in Proposition 1.10 (i.e. of size less than the size of D). These fibrations are locally trivial.

If D_1 is obtained from D by removing boxes on lines of length k, then the fiber of the map $\mathcal{F}'_1 \to \mathcal{F}_0 = \{\mathrm{pt}\}$ is the locally closed subset $\mathcal{K}_0(X, k)$. We then consider $X|_{E_1^\perp/E_1}$ and apply the induction hypothesis. □

Remark 1.13. *This result may remind the reader of results of Spaltenstein [Spa82] and Van Leuwen [vLe89] for classical (types A, B, C and D) Lie algebras. However in the Lie algebra setting the dimensions of the locally closed subsets obtained by admissible slicing are constant. The closure of these locally closed subsets are therefore the irreducible components of the Springer fiber. This does not happen in our situation.*

Example 1.14. *Let X be a nilpotent element in $\mathfrak{osp}(5,4)$ with associated diagram $D(X)$ as follows. The admissible slicings of $D(X)$ are:*

$$D(X) = \begin{ytableau} 0 & 1 \\ 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{ytableau} \supset \begin{ytableau} 1 \\ 1 \\ 0 & 1 & 0 & 1 & 0 \end{ytableau} \supset \begin{ytableau} 0 & 1 & 0 & 1 & 0 \end{ytableau} \supset \begin{ytableau} 1 & 0 & 1 \end{ytableau} \supset \begin{ytableau} 0 \end{ytableau},$$

$$D(X) = \begin{ytableau} 0 & 1 \\ 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{ytableau} \supset \begin{ytableau} 0 & 1 \\ 1 & 0 \\ 1 & 0 & 1 \end{ytableau} \supset \begin{ytableau} 0 \\ 0 \\ 1 & 0 & 1 \end{ytableau} \supset \begin{ytableau} 1 & 0 & 1 \end{ytableau} \supset \begin{ytableau} 0 \end{ytableau},$$

$$D(X) = \begin{ytableau} 0 & 1 \\ 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{ytableau} \supset \begin{ytableau} 0 & 1 \\ 1 & 0 \\ 1 & 0 & 1 \end{ytableau} \supset \begin{ytableau} 0 \\ 0 & 1 \\ 1 & 0 \end{ytableau} \supset \begin{ytableau} 0 \\ 1 \\ 1 \end{ytableau} \supset \begin{ytableau} 0 \end{ytableau}.$$

The fiber (X, B_X) is then an union of three components $(X, B_X(\overline{D}))$ of respective dimensions 2, 1, 1. To determine the irreducible components of the fiber, we have to find the locally closed subsets of dimension 1 belonging to the closure of the locally closed subset of dimension 2.

Example 1.15. *Let $X \in \mathcal{N}_1$ such that $D(X)$ is a hook. One easily checks that the locally closed subsets of $\pi^{-1}(X)$ are equidimensional, therefore B_X is equidimensional with irreducible components indexed by $\mathcal{A}(D(X))$. Let p be an even integer, the dimension of B_X is equal to:*

- $\frac{p^2}{2}$ *if $D(X)$ has an even line of length $4n+1-2p$ and $2p$ lines of length 1 (p even, p odd),*
- $\frac{p(p-2)}{2}+1$ *if $D(X)$ has an even line of length $4n+1-2p$ and $2p-2$ lines of length 1 (p odd, $p-2$ even),*
- $\frac{p(p-2)}{2}$ *if $D(X)$ has an odd line of length $4n+3-2p$ and $2p-2$ lines of length 1 (p even, $p-2$ odd),*
- $\frac{p^2}{2}$ *if $D(X)$ has an odd line of length $4n+1-2p$ and $2p$ lines of length 1 (p even, p odd).*

2 Connectedness of the fibers?

In this section we prove Theorem 0.2 *i.e.* we determine for which element $X \in \mathcal{N}_1$, the fiber $\pi^{-1}(X)$ is connected. Recall that $\mathcal{O}_1$ is the odd nilpotent subregular orbit *i.e.* of codimension 1 in $\mathfrak{osp}(2n+1,2n)$ (the fiber over this orbit is disconnected, see Proposition 3.1).

Theorem 2.1. *Let $X \in \mathcal{N}_1$, the fiber $\pi^{-1}(X)$ is connected if and only if $X \notin \mathcal{O}_1$.*

Proof : We proceed by induction on the size of the diagram associated to X and use the sequence of morphisms described in the proof of Theorem 1.12:

$$\mathcal{F}_n \to \mathcal{F}'_n \to \mathcal{F}_{n-1} \to \cdots \mathcal{F}_1 \to \mathcal{F}'_1 \to \mathcal{F}_0 = \{\text{pt}\}.$$

Let K_0 be the closure of $\mathcal{K}_0(X,1)$. The map $p : \pi^{-1}(X) \to \mathcal{F}'_1$ takes values in K_0 and is surjective. Let $\underline{D}$ be an admissible slicing of D, then $p(B_X(\underline{D}))$ is locally closed in K_0. If we consider every admissible slicing we obtain a stratification of K_0.

We prove that every point of a locally closed subset is connected, by a curve, to a point of the special locally closed subset i.e. admiting the most special admissible slicing (this slicing is obtained by removing, when there is a choice, boxes on the longest lines). We then prove that the fiber is connected if the corresponding orbit is not $\mathcal{O}_1$.

We first reduce to the case where $E_1 \in K_0$ is in the smallest stratum i.e. $E_1 \subset \mathrm{Ker}X \cap \mathrm{Im}X^{k-1}$ for k the maximal length of an even line of D. By Remark 1.8, as soon as $\dim K_0 > 0$, the closed subset K_0 is connected. Therefore, the surjectivity of p on K_0 gives us a curve in $\pi^{-1}(X)$ such that the image of the generic point is any point of K_0 (for example E_1) while the image of the special one is a point of $\mathbb{P}(\mathrm{Ker}X \cap \mathrm{Im}X^{k-1})$. Remark also (see Remark 1.8 again), that $\dim K_0 = 0$ only for the orbit $\mathcal{O}_1$.

Using this argument recursively, we are reduced to proving that if $\underline{D}$ is the special slicing described above, the locally closed subset $B_X(\underline{D})$ is connected.

Let $\underline{D}$ be the special admissible slicing of D. To choose E_1, we choose an isotropic element of $\mathbb{P}(\mathrm{Ker}X \cap \mathrm{Im}X^{k-1} \cap V_0)$ where k is the maximal length of an even line. If in D there is an even line of length at least 2, then the quadratic form φ restricted to $\mathrm{Ker}X \cap \mathrm{Im}X^{k-1} \cap V_0$ is trivial and we choose any point of $\mathbb{P}(\mathrm{Ker}X \cap \mathrm{Im}X^{k-1} \cap V_0)$ which is connected. If not then we have only even lines of length 1. We cannot have only one such line (otherwise we would be in $\mathfrak{osp}(1,0)$!) and if there are at least 3 of them, the set of isotropic elements of $\mathbb{P}(\mathrm{Ker}X \cap \mathrm{Im}X^{k-1} \cap V_0)$ is a quadric of dimension at least 1 therefore connected. It remains the case of exactly two even lines of length 1, we notice that the diagram is then the diagram associated to $\mathcal{O}_1$

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 0 \\ \cline{1-1} 0 \\ \hline 1&0&1&\cdots&1&0&1 \\ \hline\end{array}\,,$$

which is not allowed. Next we determine F_1. To do this we take an element in a projective space (the isotropic condition is always verified).

We now have to verify by induction that the diagram corresponding to $\mathcal{O}_1$ can appear in the special admissible slicing $\underline{D}$ of D only if D itself corresponds to $\mathcal{O}_1$. If the diagram corresponding to $\mathcal{O}_1$ is one of the D_i of $\underline{D}$ for $i > 0$, then D_{i-1} has one of the following two forms:

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 0 \\ \cline{1-1} 0 \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \hline 1&0&1&\cdots&1&0&1 \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|c|c|c|}\cline{1-2} 0&1 \\ \cline{1-2} 1&0 \\ \hline 1&0&1&\cdots&1&0&1 \\ \hline\end{array}\,.$$

We notice that the length of the first line is odd, it has then to be at least 3 which means that the choice of D_i was not the most special, a contradiction. $\square$

3 Fibers for the orbits $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$

We study in this section the fibers of π above the non dense orbits of maximal dimension $\mathcal{O}_i$ with $\mathrm{Codim}\mathcal{O}_i = i \in [1,3]$. The diagram of the orbit $\mathcal{O}_1$ is given in the previous proof.

Proposition 3.1. *For $X \in \mathcal{O}_1$, the fiber $\pi^{-1}(X)$ is the disjoint union of two points.*

Proof : The decomposition in locally closed subsets has only one element. Moreover, this locally closed subset is of dimension 0 and has two connected components. $\square$

The diagrams of $\mathcal{O}_2$ and $\mathcal{O}_3$ are as follows (the first lines have length $4n-3$):

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 \\ \hline \end{array}.$$

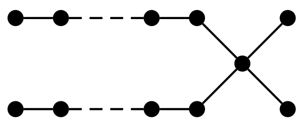
Proposition 3.2. *Let X be in the orbit $\mathcal{O}_2$ and assume that $n \geq 2$.*

($\imath$) The fiber $\pi^{-1}(X)$ is non-reduced everywhere.

($\imath\imath$) The reduced fiber, denoted by $\pi^{-1}(X)_{red}$, is the union of $2n-1$ irreducible components $(C_i)_{i\in[1,2n-1]}$, all isomorphic to $\mathbb{P}^1$, such that

- *the components $(C_{2i-1})_{i\in[1,n-2]}$ (resp. $(C_{2i})_{i\in[1,n-2]}$) form a chain that meets transversally the component C_{2n-3} in x (resp. in y distinct from x),*
- *the components C_{2n-2} and C_{2n-1} meet transversally the component C_{2n-3} in two distinct points (also distinct from x and y).*

The dual graph of $\pi^{-1}(X)_{red}$ is then the following (the left branches have length $n-2$).



Proof : ($\imath$) To find the one dimensional subspaces in $\mathrm{Ker}X \cap V_0$, we have to look for the isotropic points of $\mathbb{P}(\mathrm{Ker}X \cap V_0)$. According to Fact 1.3, the vector space $\mathrm{Ker}X \cap V_0$ has dimension 2 and the quadratic form has rank one, the unique solution is then a double point and the fiber is not reduced.

($\imath\imath$) Let us consider the decomposition into locally closed subsets, obtained in Theorem 1.12. Let $\underline{D}$ be an admissible slicing of D, the diagram associated to $\mathcal{O}_2$. We have $D_0 = D$ and D_1 has three lines: two odd, one of length $4n-5$, the other of length 3 and one even of length 1.

There are two cases for D_2. Let us denote by D_2^g the general one and by D_2^s the special one. D_2^g has three lines: one odd of length $4n-5$, two even of length 1. The diagram D_2^s has three lines: one even of length $4n-7$, another odd of length 3 and the last even of length one.

$$D_1 = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 \\ \hline 1 & 0 & 1 \\ \hline 1 & 0 & 1 & \cdots & 1 & 0 & 1 \\ \hline \end{array} \qquad D_2^g = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 \\ \hline 0 \\ \hline 1 & 0 & 1 & \cdots & 1 & 0 & 1 \\ \hline \end{array} \qquad D_2^s = \begin{array}{|c|c|c|c|c|c|}\hline 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & \cdots & 1 & 0 \\ \hline \end{array}.$$

In the case D_2^g, we have an unique choice for D_3 (one line of length $4n-5$). The diagrams D_i for $i \geq 3$ are then fixed (they have alternatively one odd line or one even line). In the case D_2^s, we recognize the diagram associated to the orbit $\mathcal{O}_2$ in $\mathfrak{osp}(2n-1, 2n-2)$. An easy induction on n proves that there are n admissible slicings.

We proceed by induction on n to prove the proposition. To start the induction, we study $\mathfrak{osp}(7,6)$. From the above, there are three admissible slicings. We denote them by $\underline{D}$, $\underline{D}'$ and $\underline{D}''$. Let us now describe explicitly the associated locally closed subsets. To do this, we fix a representative (u, u^*) of $\mathcal{O}_2$ such that the matrix of $u \in \mathrm{Hom}(V_0, V_1)$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

in the bases $(e_i)_{i\in[1,7]}$ and $(f_i)_{i\in[1,6]}$ of V_0 and V_1 such that $(e_i, e_j) = \delta_{i,8-j}$ and $(f_i, f_j) = \delta_{i,7-j}$.

The locally closed subsets are

$$B_X(\underline{D}) = \left\{ \begin{array}{l} \langle e_1\rangle \subset \langle e_1, \alpha e_2+\beta e_\pm\rangle \subset \langle e_1,e_2,e_\pm\rangle \\ \langle \alpha f_1+\beta f_3\rangle \subset \langle f_1,f_3\rangle \subset \langle f_1,f_2,f_3\rangle \end{array}, \ [\alpha:\beta]\in\mathbb{P}^1,\ \beta\neq 0\right\}$$
$$B_X(\underline{D}') = \left\{ \begin{array}{l} \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1,e_2,e_\pm(\gamma,\delta)\rangle \\ \langle f_1\rangle \subset \langle f_1,\gamma f_2+\delta f_3\rangle \subset \langle f_1,f_2,f_3\rangle \end{array}, \ [\gamma:\delta]\in\mathbb{P}^1,\ \frac{\gamma^2}{2}+\delta^2\neq 0\right\}$$
$$B_X(\underline{D}'') = \left\{ \begin{array}{l} \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1,e_2,e_\pm(\gamma,\delta)\rangle \\ \langle f_1\rangle \subset \langle f_1,\gamma f_2+\delta f_3\rangle \subset \langle f_1,\gamma f_2+\delta f_3, f_{\gamma,\delta}(\zeta,\eta)\rangle \end{array}, \frac{\gamma^2}{2}+\delta^2 = 0,\ [\zeta:\eta]\in\mathbb{P}^1\right\}$$

where $e_\pm = e_4 \pm \frac{\sqrt{2}}{2}(e_3-e_5)$ and $e_\pm(\gamma,\delta) = \sqrt{2}(\frac{\gamma}{2}(e_3+e_5)+\delta e_4) \pm \sqrt{\frac{\gamma^2}{2}+\delta^2}(e_3-e_5)$. Notice that the two values $e_\pm(\gamma,\delta)$ are equal when $\frac{\gamma^2}{2}+\delta^2=0$. We have $f_{\gamma,\delta}(\zeta,\eta) = \zeta(\frac{\gamma}{2}f_5+\mu f_4)+\eta f_3$. Let us study the closures of these locally closed subsets.

If we project $B_X(\underline{D})$ on the grassmannian $\mathbb{G}_Q(2,V_0)$ of totally isotropic subspaces of dimension 2 of V_0, its image is the union of two lines meeting in a point with the intersection point removed. The locally closed subset $B_X(\underline{D})$ is not connected and its closure is made of two irreducible components C_1 and C_2.

It is obvious that $B_X(\underline{D}'')$ is closed and its image by the projection onto $\mathbb{G}_Q(2,V_0)$ is made of two points, thus $B_X(\underline{D}'')$ has two connected components, isomorphic to $\mathbb{P}^1$, C_4 and C_5.

Finally, we can construct an isomorphism between $B_X(\underline{D}')$ and the conic of isotropic points in $\mathbb{P}(\langle e_3,e_4,e_5\rangle)$ with two points removed. Indeed, an element $e(a,b,c) = a(e_3+e_5)+be_4+c(e_3-e_5)$ belongs to this conic if and only if $2a^2+b^2+2c^2=0$. Set $a=\frac{\sqrt{2}}{2}\gamma$ and $b=\sqrt{2}\delta$, we have $c=\pm\sqrt{\frac{\gamma^2}{2}+\delta^2}$. We obtain the following description

$$B_X(\underline{D}') = \left\{ \begin{array}{l} \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1,e_2,e(a,b,c)\rangle \\ \langle f_1\rangle \subset \langle f_1,2af_2+bf_3\rangle \subset \langle f_1,f_2,f_3\rangle \end{array}, 2a^2+b^2+2c^2=0,\ c\neq 0\right\}.$$

The closure of $B_X(\underline{D}')$ is isomorphic to $\mathbb{P}^1$ and gives us the component C_3. The intersections between the distinct components come from previous descriptions.

Let us go back to the general case. We know that there are n admissible slicings. We denote them by $\underline{D}^i$ for $i\in[1,n]$. If $\underline{D}^1$ is the general one, we know that the slicings $\underline{D}^i$ for $i>1$ are all slicings of the diagram associated to the orbit $\mathcal{O}_2$ in $\mathfrak{osp}(2n-1,2n-2)$. The union of the corresponding locally closed subsets is isomorphic to the fiber above the orbit $\mathcal{O}_2$ in $\mathfrak{osp}(2n-1,2n-2)$. To end the proof, we only have to check, by induction on n that the locally closed subset $B_X(\underline{D}^1)$ which gives us the components C_1 and C_2 meets the fiber as expected. We proceed as in the $\mathfrak{osp}(7,6)$ case. We begin by choosing a representative u of the orbit $\mathcal{O}_2$ defined by $u(e_i)=f_{i-1}$ for $i\in[1,2n+1]\setminus\{n+2\}$ and $u(e_{n+2})=f_{n-1}$ (by convention $f_0=0$) where $(e_i)_{i\in[1,2n+1]}$ and $(f_i)_{i\in[1,2n]}$ are the bases of V_0 and V_1 respectively such that $(e_i,e_j)=\delta_{i,2n+2-j}$ and $(f_i,f_j)=\delta_{i,2n+1-j}$. We can describe the locally closed subsets $B_X(\underline{D}^i)$:

$$B_X(\underline{D}^1) = \left\{ \begin{array}{l} \langle e_1\rangle \subset \langle e_1, \alpha e_2+\beta e_\pm\rangle \subset \langle e_1,e_2,e_\pm\rangle \subset\cdots\subset \langle e_1,e_2,\cdots,e_{n-1},e_\pm\rangle \\ \langle \alpha f_1+\beta f_n\rangle \subset \langle f_1,f_n\rangle \subset \langle f_1,f_2,f_n\rangle \subset\cdots\subset\langle f_1,f_2,\cdots,f_n\rangle \end{array}, \ \beta\neq 0\right\}$$
$$B_X(\underline{D}^2) = \left\{ \begin{array}{l} \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1,e_2,\gamma e_3+\delta e_\pm\rangle \subset\cdots\langle e_1,e_2,\cdots,e_{n-1},e_\pm\rangle \\ \langle f_1\rangle \subset \langle f_1,\gamma f_2+\delta f_3\rangle \subset \langle f_1,f_2,f_3\rangle \subset\cdots\subset\langle f_1,f_2,\cdots,f_n\rangle \end{array}, \ \delta\neq 0\right\},$$

where $e_\pm = e_{n+1}\pm\frac{\sqrt{2}}{2}(e_n-e_{n+2})$. The locally closed subsets $B_X(\underline{D}^1)$ and $B_X(\underline{D}^2)$ have two connected components and their closures have two irreducible components denoted respectively by C_1, C_2 for $B_X(\underline{D}^1)$ and C_3 C_4 for $B_X(\underline{D}^2)$. These components intersect each other as predicted.

For the sets $B_X(\underline{D}^i)$ with $i \geq 2$, we only check that the three first subspaces of the complete flag of V_0 are always $\langle e_1 \rangle \subset \langle e_1 e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle$. In particular, the closures of the locally closed subsets $B_X(\underline{D}^i)$ for $i \geq 2$ do not meet the closure of $B_X(\underline{D}^1)$. $\square$

3.1 Orbit $\mathcal{O}_3$

Proposition 3.3. *Let $n \geq 2$ and $X \in \mathfrak{osp}(2n+1, 2n)$ be in the orbit $\mathcal{O}_3$. The reduced fiber $\pi^{-1}(X)_{red}$ is the union of $2n-2$ irreducible components $(C_i)_{i \in [1,n-1]}$ and $(S_i)_{i \in [1,n-1]}$ such that*

- *the components $(C_i)_{i \in [1,n-1]}$ are isomorphic to $\mathbb{P}^1$,*
- *the components $(S_i)_{i \in [1,n-1]}$ are isomorphic to the blow-up of $\mathbb{P}^2$ in two distinct points,*
- *the components $(C_i)_{i \in [1,n-1]}$ form a chain,*
- *the components $(S_i)_{i \in [1,n-1]}$ form a chain, two components intersect along a $\mathbb{P}^1$,*
- *the two chains intersect each other in a point on $C_{n-1} \cap S_{n-1}$.*

Proof : We proceed by induction as in the previous proposition. We first study the $\mathfrak{osp}(7,6)$ case. There are five admissible slicings $(\underline{D}^i)_{i \in [1,5]}$ of D. We use the bases $(e_i)_{i \in [1,7]}$ and $(f_i)_{i \in [1,6]}$ defined in the proof of the previous proposition to choose a representative of the orbit u, whose matrix is the following one

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The locally closed subsets are:

$$\begin{aligned}
B_X(\underline{D}^1) &= \left\{ \begin{array}{l} \langle ae_1 + be_3 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle \alpha(af_1 + bf_3) + \beta f_4 \rangle \subset \langle f_1, \alpha b f_3 + \beta f_4 \rangle \subset \langle f_1, f_2, \alpha b f_3 + \beta f_4 \rangle \end{array}, \ b \neq 0 \right\} \\
B_X(\underline{D}^2) &= \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, \alpha e_2 + \beta e_5 \rangle \subset \langle e_1, e_2, e_5 \rangle \\ \langle \alpha f_1 + \beta f_4 \rangle \subset \langle f_1, f_4 \rangle \subset \langle f_1, f_2, f_4 \rangle \end{array}, \ \beta \neq 0 \right\} \\
&\coprod \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle \alpha f_1 + \beta f_4 \rangle \subset \langle f_1, f_4 \rangle \subset \langle f_1, f_2, f_4 \rangle \end{array}, \ \beta \neq 0 \right\} \\
B_X(\underline{D}^3) &= \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, ce_2 + de_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle f_1 \rangle \subset \langle f_1, \gamma(cf_2 + df_3) + \delta f_4 \rangle \subset \langle f_1, f_2, \gamma d f_3 + \delta f_4 \rangle \end{array}, \ d \neq 0 \right\} \\
B_X(\underline{D}^4) &= \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, \gamma^2 e_3 - 2\gamma\delta e_4 - 2\delta^2 e_5 \rangle \\ \langle f_1 \rangle \subset \langle f_1, \gamma f_2 + \delta f_4 \rangle \subset \langle f_1, f_2, f_4 \rangle \end{array}, \ \delta \neq 0 \right\} \\
&\coprod \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle f_1 \rangle \subset \langle f_1, \gamma f_2 + \delta f_4 \rangle \subset \langle f_1, f_2, f_4 \rangle \end{array}, \ \delta \neq 0 \right\} \\
B_X(\underline{D}^5) &= \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle f_1 \rangle \subset \langle f_1, f_2 \rangle \subset \langle f_1, f_2, \zeta f_3 + \eta f_4 \rangle \end{array}, \ \delta \neq 0 \right\}.
\end{aligned}$$

The closure of $B_X(\underline{D}^1)$ has in its boundary the two following lines

$$\left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle \alpha f_1 + \beta f_4 \rangle \subset \langle f_1, f_4 \rangle \subset \langle f_1, f_2, f_4 \rangle \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle f_1 \rangle \subset \langle f_1, xf_3 + yf_4 \rangle \subset \langle f_1, f_2, xf_3 + yf_4 \rangle \end{array} \right\}.$$

The second one is an exceptional divisor and belongs to the closure of $B_X(\underline{D}^3)$ when c vanishes. The second exceptional divisor is obtained by putting $\beta = 0$ in $B_X(\underline{D}^1)$. We can see then that the second component of $B_X(\underline{D}^2)$ is in the closure of $B_X(\underline{D}^1)$. In the same way, the closure of $B_X(\underline{D}^3)$ has in its boundary two lines:

$$\left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle f_1 \subset \langle f_1, \gamma f_2 + \delta f_4 \rangle \subset \langle f_1, f_2, f_4 \rangle \rangle \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ \langle f_1 \rangle \subset \langle f_1, f_2 \rangle \subset \langle f_1, f_2, x f_3 + y f_4 \rangle \end{array} \right\}.$$

The second line is a component of $B_X(\underline{D}^5)$. The rest of the proposition follows from this description. For the general case, we proceed as for the orbit $\mathcal{O}_2$. We omit the details of the proof. □

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